International Journal of Theoretical Physics, Vol. 35, No. 7, 1996

Unsharp Observables and Their Joint Measurement

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Received July 19, 1995

It is shown that in the double-slit experiment, which is an unsharp path determination if represented by a generalized Luder operation, the interference term in the probability expression exactly corresponds to one of the marginals representing an unsharp interference observable in the realistic joint measurement presented by Busch. A complicated arrangement is presented to show a nontrivial joint triple measurement for spin-1/2 observables

1. INTRODUCTION

The complementarity of particle and wave aspects of quantum phenomena is the most striking feature of quantum mechanics, as revealed in the double-slit experiment, in which one may either observe a path or an interference pattern, but it is impossible to observe simultaneously a particle and a wave. It has been shown that this strict complementarity can be relaxed in the framework of an unsharp formalism of observables. Wooters and Zurek (1979) showed that one may observe the path with 95% confidence and still obtain very visible interference effects. Mittelstaedt *et al.* (1987) presented a realization of this idea by means of a photon split-beam experiment, using a Mach–Zender interferometer.

Busch (1985, 1986) presented a mathematical framework which provides joint measurement for complementary observables in two-dimensional Hilbert space. For a two-dimensional quantum system, noncommutivity of (sharp) observables is equivalent to their complementarity. Only in the POV measure formalism of observables can two complementary observables be simultaneously measurable when the intrinsic unsharpness introduced into the observables is sufficiently large. In the case of the much-debated issue of the Bell-CHSH inequality, it has been shown that in quantum mechanics inequalities

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0020-7748/96/0700-1279$09.50/0

can be satisfied if the observables are sufficiently unsharp. In other words, this implies that the joint distribution of some observables not allowed by standard quantum mechanics may be possible in the POV formalism of quantum mechanics (Busch, 1985, 1986; Kar and Roy, 1995).

Realistic joint measurements in the case of the double-slit experiment as well as for spin-1/2 observables have been presented. The basic idea of these experiments is to enlarge the state space of the system—as one actually does in any experiment by combining the system with the measuring device. All arrangements are composed of elements like beam splitters and prisms which are describable as ideal filters, and mathematically they are presented by projection operators, but their complicated arrangements give rise to effects representing various joint observables (Busch, 1987; Busch and Schroek, 1989; de Muynck and Martens, 1990).

Here we show that in the double-slit experiment, which is an unsharp path determination if represented by a generalized Luder operation, the interference term in the probability expression for the position of the particle on the screen corresponds to one of the marginals (representing an unsharp interference observable) of the realistic joint measurement for the double-slit experiment described by Busch. Realistic joint measurement for two spin-1/2 observables has been presented by using a split-beam experiment. In principle this experiment is realizable with neutron beams. This example can also be interpreted as a joint triple measurement for three complementary spin-1/2 observables. But this is a very special case, as four out of eight effects in the range of triple observables ($\{F_{ijk}\}$, $i = 1, \bar{1}, j = 2, \bar{2}, k = 3, \bar{3}$) are zero. In this paper we present a slightly more complicated arrangement (enlarging the state space by splitting the beam once again), which provides eight nonzero effects in the range of the triple observables. By suitably choosing the parameters of the experiment, we can select one of the marginals which measures spin (unsharply) in a desired direction.

2. FORMALISM

Any linear operator A on two-dimensional Hilbert space can be written in the form

$$A = \alpha_0 I + \alpha \cdot \sigma, \qquad (\alpha_0, \alpha) \in \mathcal{C}^4 \tag{1}$$

In particular, effects on the two-dimensional Hilbert space F ($0 \leq F \leq I$) can be written as

$$F = \tfrac{1}{2}\gamma(I + \lambda \cdot \sigma) \tag{2}$$

with $0 \leq \gamma$, $\|\lambda\| \leq 1$,

$$F = \gamma E(\lambda) \tag{3}$$

where

$$E(\lambda) = \tfrac{1}{2}(I + \lambda \cdot \sigma) \tag{4}$$

For $\|\lambda\| = 1$, $E(\lambda)$ is a projection operator and it represents ordinary sharp spin properties and will be denoted by $P(\hat{\lambda})$. The requirement that the 'spin-up' and 'spin-down' results should be complementary to each other, i.e., $\overline{F} = I - F = F(-\lambda, \gamma)$, implies $\gamma = 1$, i.e., only $E(\lambda)$ may be spin properties.

The spectral decomposition of $E(\lambda)$ is

$$E(\lambda) = \tfrac{1}{2}(1 + \|\lambda\|)P(\hat{\lambda}) + \tfrac{1}{2}(1 - \|\lambda\|)P(-\hat{\lambda}) \tag{5}$$

The eigenvalues have the following interpretation:

$$r = \tfrac{1}{2}(1 + \|\lambda\|) > \tfrac{1}{2}$$

is the reality degree of property $P(\hat{\lambda})$; and

$$u = \tfrac{1}{2}(1 - \|\lambda\|) < \tfrac{1}{2}$$

is the unsharpness of property $P(\hat{\lambda})$.

The generalized Luder measurement (Busch, 1986) ϕ_L of the unsharp spin property $E(\lambda)$ is defined as

$$\phi_L D = E(\lambda)^{1/2} D E(\lambda)^{1/2} + E(-\lambda)^{1/2} D E(-\lambda)^{1/2} \tag{6}$$

where D is the initial state of the system to be measured.

As mentioned in the Introduction, any two spin-1/2 observables are complementary. For unsharp cases, however, they may allow joint measurement (Busch, 1985). In general, two effects F_1 and F_2 are coexistent if and only if there exist four effects $(F_{12}, F_{1\bar{2}}, F_{\bar{1}2}, F_{\bar{1}\bar{2}})$ which satisfy (Kraus, 1983)

$$F_1 = F_{12} + F_{1\bar{2}}, \qquad F_2 = F_{12} + F_{\bar{1}2} \tag{7}$$

$$F_{\bar{1}} = F_{\bar{1}2} + F_{\bar{1}\bar{2}}, \qquad F_{\bar{2}} = F_{1\bar{2}} + F_{\bar{1}\bar{2}}$$

Applying this criterion for spin, it has been shown that two unsharp spin properties $E(\lambda_1)$ and $E(\lambda_2)$ of the form (4) are coexistent if and only if

$$\tfrac{1}{2}\|\lambda_1 + \lambda_2\| + \tfrac{1}{2}\|\lambda_1 - \lambda_2\| \leq 1 \tag{8}$$

so any pair of directions $\hat{\lambda}_1$ and $\hat{\lambda}_2$ can be made coexistent by introducing a sufficiently large unsharpness. Similarly the triple $E(\lambda_1)$, $E(\lambda_2)$, and $E(\lambda_3)$ can be coexistent if

$$\lambda_1^2 + \lambda_2^2 + \lambda_3^2 \leq 1 \tag{9}$$

3. JOINT MEASUREMENT IN DOUBLE-SLIT EXPERIMENT

The double-slit experiment can be interpreted in terms of conditional probability (Beltrametti and Cassinelli, 1981). To switch over to unsharp path measurement in the double-slit experiment, we first describe the conditional probability interpretation of this experiment.

In the double-slit experiment the screen S_1 has two slits E_1 and E_2. Let us imagine a free particle traveling toward the screen S_1 in the direction of the x axis with constant velocity v. We are interested in the probability distribution of the position of the particle on the screen S_2 (behind the screen S_1). Let $P[\psi]$ be the density operator representing the initial state of the particle. Let $P(E)$ be the projection-valued measure that the particle is confined in the Borel set E. Let $\|P(E_i)\psi\|(i = 1, 2) \neq 0$. We are interested in the conditional probability that the y coordinate of the particle has value in the set E of the screen S_2 at time $t = \tau$, given that it was localized in the set $F_1 \cup E_2$ of the screen S_1 at time $t = 0$. We write

$$C_{E_i} = \frac{\|P(E_i)\psi\|}{\|P(E_1 \cup E_2)\psi\|}$$

and

$$\psi_{E_i} = \frac{P(E_i)\psi}{\|P(E_i)\psi\|}$$

The conditional probability is given by

$$p(P(E), t = \tau/P(E_1 \cup E_2), t = 0) = \frac{\mathrm{Tr}[P(E_1 \cup E_2)P[\psi]P(E_1 \cup E_2)U_\tau^{-1}P(E)U_\tau]}{\mathrm{Tr}[P(E_1 \cup E_2)P[\psi]]} \quad (10)$$

The intervals E_1 and E_2 are disjoint, and hence $P(E_1 \cup E_2) = P(E_1) + P(E_2)$, then

$$P(E_1 \cup E_2)\psi = \|P(E_1 \cup E_2)\psi\|(C_{E_1}\psi_{E_1} + C_{E_2}\psi_{E_2}) \quad (11)$$

Then

$$p(P(E), t = \tau/P(E_1 \cup E_2), t = 0 = \langle C_{E_1}\psi_{E_1} + C_{E_2}\psi_{E_2} | U_\tau^{-1}P(E)U_\tau | C_{E_1}\psi_{E_1} + C_{E_2}\psi_{E_2}\rangle \quad (12)$$

The interference term in this expression for the probability distribution is

$$2C_{E_1}C_{E_2} \int U_\tau\psi_{E_1}U_\tau\psi_{E_2}\, dy$$

which is different from zero for $\tau \neq 0$. In fact, although ψ_{E_1} and ψ_{E_2} have disjoint supports, U_τ (the unitary operator) spreads them over the whole y axis.

On the other hand, if the path determination is done sharply, the conditional probability will be given by

$$p(P(E), t = \tau / P(E_1) \text{ or } P(E_2), t = 0)$$
$$= \frac{\mathrm{Tr}[\{P(E_1)P[\psi]P(E_1) + P(E_2)P[\psi]P(E_2)\}U_\tau^{-1}P(E)U_\tau]}{\mathrm{Tr}[(P(E_1) + P(E_2))P[\psi]]}$$
$$= |C_1|^2 \int |U_\tau \psi_{E_1}|^2 \, dy + |C_2|^2 \int |U_\tau \psi_{E_2}|^2 \, dy \tag{13}$$

So there is no interference term.

For an unsharp path determination of a microparticle in the double-slit experiment we consider the two effects

$$F_1 = (1 - \epsilon)P(E_1) + \epsilon P(E_2), \qquad F_2 = (1 - \epsilon)P(E_2) + \epsilon P(E_1) \tag{14}$$

with $0 \leq \epsilon \leq 1/2$. The F_i corresponds to an unsharp measurement corresponding to localization in the sets E_i.

In the last expression for the conditional probability distribution if we replace $P(E_1)$ by F_1 and $P(E_2)$ by F_2 and the Luder operation by the generalized Luder operation, we get

$$p(P(E), t = \tau / F_1 \text{ or } F_2, t = 0)$$
$$= \frac{\mathrm{Tr}[\{F_1^{1/2}P[\psi]F_1^{1/2} + F_2^{1/2}P[\psi]F_2^{1/2}\}U_\tau^{-1}P(E)U_\tau]}{\mathrm{Tr}[(F_1 + F_2)P[\psi]]} \tag{15}$$

Writing the spectral expansion of F_1 and F_2, we find the interference term in the last probability expression

$$4\sqrt{(1 - \epsilon)\epsilon}\, C_{E_1} C_{E_2} \int U_\tau \psi_{E_1} U_\tau \psi_{E_2} \, dy$$

Obviously for $\epsilon = 0$, i.e., for a sharp path determination, there is no interference term and for $\epsilon = 1/2$, i.e., for a randomized path determination, the interference is full. It has been shown that the double-slit experiment can be fully described in terms of two-dimensional Hilbert space and the path and interference observables are described by two unit vectors orthogonal to each other in the Poincaré sphere. In this representation our (unsharp) path observable F_1 will be given by

$$F_1 = \tfrac{1}{2}[I + (2\epsilon - 1)z \cdot \sigma] \tag{16}$$

provided the unit vector z represents a standard path observable on the Poincaré sphere. In a realistic joint measurement of path and interference in the double-slit experiment described by Busch (1987) the two marginals E_1 and E_2 (representing unsharp path and interference observables, respectively) out of four joint observables are given by

$$E_1 = \tfrac{1}{2}[I + (2\gamma - 1)z \cdot \sigma], \qquad E_2 = \tfrac{1}{2}[I + \sqrt{\gamma(2\gamma - 1)}x \cdot \sigma] \tag{17}$$

where γ is the transparency of the mirror employed in the split-beam experiment. Now it is interesting to note that the blurred interference term in the last expression for the probability distribution exactly corresponds to the unsharp interference observable E_2 when $\epsilon = \gamma$. For $\epsilon = \gamma = 1/2$, E_2 is a projection operator representing a sharp interference observable.

4. REALISTIC JOINT TRIPLE MEASUREMENT FOR SPIN-1/2 OBSERVABLES

In this section we present a more generalized model where all eight effects in the range of the triple observable are nonzero and one of the marginals can be chosen in an arbitrary direction depending on the parameters of the apparatus. To achieve this, we have to find an arrangement yielding eight possible mutually exclusive outcomes. The basic idea is to enlarge the state space of the system, as is done in any experiment, by combining the system with the measuring device.

This complicated arrangement is shown in Fig. 1. The incoming beam of spin-1/2 particles is split according to orthogonal polarization states ϕ_+ (eigenstates of the prism observable) which are singled out by prism P. The two partial beams are again split by two transparent mirrors with transparency γ_1 (say). Then the two pairs of four partial beams are combined and split

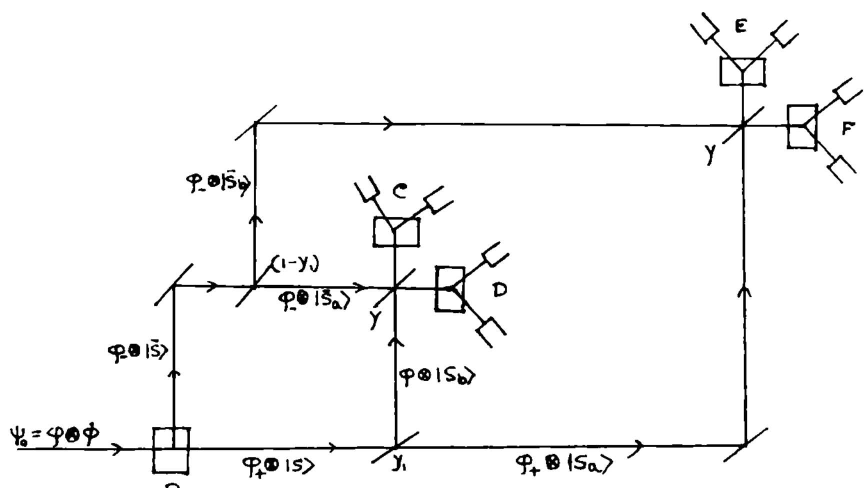

Fig. 1. Setup for a joint measurement of three spin-1/2 observables. The incoming beam is split by means of a prism P. The two partial beams are again split by two transparent mirrors with transparency y_i. Then four partial beams, taken two at a time, are recombined and sent through transparent mirrors with transparency y. The outgoing beam polarizations are analyzed by means of prisms C, D, E, and F.

again by a γ-transparent mirror into two two-polarization-measuring devices consisting of a prism of pairs of detectors (C_+, D_+) and (E_+, F_+).

Let the initial state of polarization be

$$\phi = c\phi_+ + e^{i\chi}(1 - c^2)^{1/2}\phi_- \tag{18}$$

Then the state evolution as shown in Fig. 1 is given by

$$\begin{aligned}\psi_o = \phi \otimes \varphi \to \psi &= c\phi_+ \otimes |s\rangle + e^{i\chi}(1 - c^2)^{1/2}\phi_- \otimes |\bar{s}\rangle \\ &= c\phi_+ \otimes [\gamma_1^{1/2}|s_a\rangle + (1 - \gamma_1)^{1/2}|s_b\rangle] \\ &\quad + e^{i\phi}(1 - c^2)^{1/2}\phi_- \otimes [\gamma_1^{1/2}|\bar{s}_b\rangle + (1 - \gamma_1)^{1/2}|\bar{s}_a\rangle] \\ &= (1 - \gamma_1)^{1/2}[c\phi_+ \otimes |s_b\rangle + e^{i\phi}(1 - c^2)^{1/2}\phi_- \otimes|\bar{s}_a\rangle] \\ &\quad + \gamma_1^{1/2}[c\phi_+ \otimes |s_a\rangle + e^{i\phi}(1 - c^2)^{1/2}\phi_- \otimes |\bar{s}_b\rangle]\end{aligned} \tag{19}$$

Now the four path states $|s_a\rangle$, $|s_b\rangle$, $|\bar{s}_a\rangle$, and $|\bar{s}_b\rangle$ are developed in terms of the orthogonal detector path states $|v\rangle$, $|\bar{v}\rangle$, $|w\rangle$, and $|\bar{w}\rangle$:

$$\begin{aligned}|v\rangle &= \sqrt{(1 - \gamma)}[|s_b\rangle + \sqrt{\gamma}|\bar{s}_a\rangle] \\ |\bar{v}\rangle &= \sqrt{\gamma}[|s_b\rangle - \sqrt{(1 - \gamma)}|\bar{s}_a\rangle] \\ |w\rangle &= \sqrt{(1 - \gamma)}[|s_a\rangle + \sqrt{\gamma}|\bar{s}_b\rangle] \\ |\bar{w}\rangle &= \sqrt{\gamma}[|s_a\rangle - \sqrt{(1 - \gamma)}|\bar{s}_b\rangle]\end{aligned} \tag{20}$$

Then using the last equation, we can write

$$\begin{aligned}\psi = &\sqrt{(1 - \gamma_1)}[\pi_c\phi \otimes |v\rangle + \pi_d\phi \otimes |\bar{v}\rangle] \\ &+ \sqrt{\gamma_1}[\pi_c\phi \otimes |w\rangle + \pi_d\phi \otimes |\bar{w}\rangle]\end{aligned} \tag{21}$$

where

$$\begin{aligned}\pi_c &= \sqrt{\gamma}P[\phi_+] - \sqrt{1 - \gamma}P[\phi_-] \\ \pi_d &= \sqrt{\gamma}P[\phi_-] + \sqrt{1 - \gamma}P[\phi_+]\end{aligned} \tag{22}$$

Here $P[\cdot]$ is a one-dimensional projection operator projecting on the vector in the third bracket.

On the preparation ψ let one measure the projection-valued observables constituted by

$$\bar{C}_i \otimes P[w], \quad \bar{D}_j \otimes P[v], \quad \bar{E}_k \otimes P[\bar{w}], \quad \bar{F}_l \otimes P[\bar{v}] \qquad (i, j, k, l = +, -)$$

The above eight projections are pairwise orthogonal and thus commutative, even though $\bar{C}_i$, $\bar{D}_j$, etc., are not generally commutative.

The probabilities of the outcomes C_i, D_j, etc., are given by

$$\langle\psi|\bar{C}_i \otimes P[w]|\psi\rangle = \sqrt{\gamma_1}\langle\pi_c\phi|\bar{C}_i|\pi_c\phi\rangle = \langle\phi|C_i|\phi\rangle \tag{23}$$

where the effect C_i is given by

$$C_i = \gamma_1 \pi_c^\dagger \overline{C}_i \pi_c$$

Similarly,

$$\begin{aligned} D_j &= (1 - \gamma_1)\pi_c^\dagger \overline{D}_j \pi_c \\ E_k &= \gamma_1 \pi_d^\dagger \overline{E}_k \pi_d \\ F_l &= (1 - \gamma_1)\pi_d^\dagger \overline{F}_l \pi_d \end{aligned} \tag{24}$$

Let the projection operators $\overline{C}_\pm$, $\overline{D}_\pm$, etc., be given by

$$\begin{aligned} \overline{C}_\pm &= \tfrac{1}{2}[I \pm c\cdot\sigma], \qquad \overline{D}_\pm = \tfrac{1}{2}[I \pm d\cdot\sigma] \\ \overline{E}_\pm &= \tfrac{1}{2}[I \pm e\cdot\sigma], \qquad \overline{F}_\pm = \tfrac{1}{2}[I \pm f\cdot\sigma] \end{aligned} \tag{25}$$

with c, d, e, f all unit vectors. Identifying the prism observables with the polar direction of the Poincaré sphere,

$$P[\phi_+] = \tfrac{1}{2}[I + z\cdot\sigma] \tag{26}$$

and identifying all eight effects of (24) with the eight effects in the range of the triple observable F_{ijk}, we can write

$$\begin{aligned} C_+ &= \gamma_1 \tfrac{1}{2} c_0^+ [I + c_+\cdot\sigma] = F_{123} \\ C_- &= \gamma_1 \tfrac{1}{2} c_0^- [I + c_-\cdot\sigma] = F_{\bar{1}23} \\ D_+ &= (1 - \gamma_1)\tfrac{1}{2} d_0^+ [I + d_+\cdot\sigma] = F_{1\bar{2}3} \\ D_- &= (1 - \gamma_1)\tfrac{1}{2} d_0^- [I + d_-\cdot\sigma] = F_{\bar{1}23} \\ E_+ &= \gamma_1 \tfrac{1}{2} e_0^+ [I + e_+\cdot\sigma] = F_{1\bar{2}\bar{3}} \\ E_- &= \gamma_1 \tfrac{1}{2} e_0^- [I + e_-\cdot\sigma] = F_{\bar{1}2\bar{3}} \\ F_+ &= (1 - \gamma_1)\tfrac{1}{2} f_0^+ [I + f_+\cdot\sigma] = F_{12\bar{3}} \\ F_- &= (1 - \gamma_1)\tfrac{1}{2} f_0^- [I + f_-\cdot\sigma] = F_{\bar{1}\bar{2}\bar{3}} \end{aligned} \tag{27}$$

where

$$\begin{aligned} c_0^\mp &= \tfrac{1}{2}[1 \pm (2\gamma - 1)c_3] \\ c_\pm &= [2c_0^\mp]^{-1}(\mp 2\sqrt{\gamma(1 - \gamma)}c_1, \mp 2\sqrt{\gamma(1 - \gamma)}c_2, (2\gamma - 1) \pm c_3)^T \end{aligned} \tag{28}$$

We can write similar expressions for $d_\pm$, $e_\pm$, etc.

Let us now see under what conditions the marginals of all these effects can represent an unsharp spin property with the form given by

$$\tfrac{1}{2}[I + a\cdot\sigma] \qquad \text{with} \quad 0 < |a| \le 1$$

Now the marginal F_1 is given by

$$\begin{aligned} F_1 &= F_{123} + F_{12\bar{3}} + F_{1\overline{23}} + F_{1\bar{2}3} \\ &= \tfrac{1}{2}[(f_0^+ + d_0^+ + \gamma_1\{c_0^+ + e_0^+ - f_0^+ - d_0^+\})I + f_1\cdot\sigma] \end{aligned} \tag{29}$$

where

$$f_1 = [\gamma_1 c_0^+ c^+ + (1 - \gamma_1) f_0^+ f^+ + \gamma_1 e_0^+ e^+ + (1 - \gamma_1) d_0^+ d^+] \tag{30}$$

Putting in the values of $c_0^\mp$, $c^\pm$, etc., we get

$$\begin{aligned} F_1 = \tfrac{1}{4}[2 + (2\gamma - 1)(d_3 - f_3) + \gamma_1(2\gamma - 1)(c_3 + f_3 - e_3 \\ - d_3)] + \tfrac{1}{2} f_1\cdot\sigma \end{aligned} \tag{31}$$

Similarly, for the marginal effect F_2

$$\begin{aligned} F_2 = \tfrac{1}{4}[2 - (2\gamma - 1)(d_3 + f_3) + \gamma_1(2\gamma - 1)(c_3 + f_3 + e_3 \\ + d_3)] + \tfrac{1}{2} f_2\cdot\sigma \end{aligned} \tag{32}$$

where

$$f_2 = [\gamma_1 c_0^+ c^+ + (1 - \gamma_1) f_0^+ f^+ + \gamma_1 e_0^- e^- + (1 - \gamma_1) d_0^- d^-] \tag{33}$$

$$F_3 = \pi_c^\dagger \pi_c = \tfrac{1}{2}[I + (2\gamma - 1) z\cdot\sigma] \tag{34}$$

Now for F_1, F_2, and F_3 to represent unsharp spin properties, at least one of the following conditions should be satisfied:

(a) $c_3 = d_3 = e_3 = f_3 = 0$.
(b) $\gamma_1 = \tfrac{1}{2}$, $c_3 = d_3 = e_3 = f_3$.

For condition (a), the two unsharp observables F_1 and F_2 will be represented in the Poincaré sphere by vectors f_1 and f_2 which are in the xy plane.

But if condition (b) is satisfied with $c_3 \neq 0$, then one of them (here F_1) will be represented by a vector with nonzero z component, i.e., $(f_1)_z = c_3 \neq 0$. So, unlike the example presented by Busch, in our case F_1 and F_2 are not both forced to represent spin measurement along vectors in the xy plane. So this model represents a joint triple measurement of three unsharp observables whose corresponding vectors lie along the z axis, in the xy plane, and in space depending on the parameters of the apparatus employed.

5. DISCUSSION

In a double-slit experiment some kind of unsharp path determination implies the realization of a joint measurement for both path and interference observables. We use a generalized Luder operation because it disturbs the initial state minimally. It is interesting to note that the interference term in this case exactly corresponds to the (unsharp) interference observable in the realistic joint measurement in Busch (1987).

Busch showed that introduction of a relative phase shift in the two partial beams has the effect of rotating the vectors in the unsharp joint observables about the z axis. To avoid complications we do not introduce a phase shift. Finally, it should be noted that an experiment for a joint triple measurement is in principle realizable with neutron beams. An inhomogeneous magnetic field may serve as a prism, and a homogeneous magnetic field, in the case of polarized partial beams, as a mirror.

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